THE NEOCLASSICAL SOLOW-SWAN GROWTH MODEL

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Some general remarks about models and assumptions in economics:

- What is a model? A mathematical description of the economy.
- Why do we need a model? The world is too complex to describe it in every detail.
- What makes a model successful? When it is simple but effective in describing and predicting how the world works.
- A model relies on simplifying assumptions. These assumptions drive the conclusions of the model. When analyzing a model it is crucial to spell out the assumptions underlying the model.
- Realism may not outweigh the property of a good assumption.

Introduction

Useful definitions: the growth rate

\[
\text{Growth rate of output} = \frac{Y_t - Y_{t-1}}{Y_{t-1}}
\]

In continuous time:

\[
\text{Growth rate of output} = \frac{dY}{dt} = \frac{\dot{Y}}{Y}
\]

In per capita terms:

\[
\text{Growth rate of per capita output} = \frac{\dot{y}}{y} = \frac{(Y / N)}{Y / N} = \frac{\dot{Y}}{Y} - \frac{\dot{N}}{N}
\]

...and some main stylized facts in growth:
- Per capita income rises
- Growth rates between countries differ
- Capital-output ratio is steady
- Capital-labour ratio increases
- Shares of capital and labour are constant
The basic structure of the model

Aggregate production function:

\[ Y_t = F(K_t, L_t) \]

where \( Y_t \) : output, \( K_t \) : capital stock, \( L_t \) : labour force

Neoclassical assumptions on the production function:

(1) Positive and diminishing marginal returns of factor inputs:

\[
MPK_t = \frac{\partial F(K_t, L_t)}{\partial K_t} > 0, \quad MPK_t = \frac{\partial^2 F(K_t, L_t)}{\partial K_t^2} < 0
\]

\[
MPL_t = \frac{\partial F(K_t, L_t)}{\partial L_t} > 0, \quad MPL_t = \frac{\partial^2 F(K_t, L_t)}{\partial L_t^2} < 0
\]

(2) Constant returns to scale w.r.t capital and labour:

\[ F(\lambda K_t, \lambda L_t) = \lambda F(K_t, L_t) \quad \text{τοπο ι>0} \]

(3) Inada conditions:

\[
\lim_{K_t \to 0} (MPK_t) = \lim_{L_t \to 0} (MPL_t) = \infty
\]

\[
\lim_{K_t \to \infty} (MPK_t) = \lim_{L_t \to \infty} (MPL_t) = 0
\]

In per capita terms by condition (2):

\[
\frac{Y_t}{L_t} = F\left(\frac{K_t}{L_t}, 1\right) \Rightarrow y_t = f(k_t)
\]

where \( y_t = Y_t/L_t \), \( k_t = K_t/L_t \), and \( f(k_t) = F(k_t, 1) \). (Note: \( y_t \) can increase only if \( k_t \) increases over time). By the neoclassical assumptions:

\[
MPK_t = \frac{\partial F(K_t, L_t)}{\partial K_t} = \frac{\partial [L_t f(k_t)]}{\partial K_t} = f'(k_t) > 0
\]

\[
\frac{\partial MPK_t}{\partial K_t} = \frac{\partial f''(k_t)}{\partial K_t} = \frac{1}{L_t} f''(k_t) < 0 \Rightarrow f''(k_t) < 0
\]
An example: The *Cobb-Douglas* production function

$$Y = AK^a L^{1-a}$$

where $A > 0$ is a technology parameter and $0 < a < 1$.

(1) The Cobb-Douglas production function has positive and diminishing marginal production w.r.t. capital and labour:

$$\frac{\partial Y}{\partial K} = \alpha A \left( \frac{L}{K} \right)^{1-a} > 0, \quad \frac{\partial^2 Y}{\partial K^2} = -\alpha (1 - \alpha) AK^{a-2} L^{1-a} < 0$$

$$\frac{\partial Y}{\partial L} = (1 - \alpha) A \left( \frac{K}{L} \right)^{a} > 0, \quad \frac{\partial^2 Y}{\partial L^2} = -\alpha (1 - \alpha) AK^a L^{-1-a} < 0$$

(2) The Cobb-Douglas production function exhibits constant returns to scale:

$$A(\lambda K)^a (\lambda L)^{1-a} = \lambda^a \lambda^{1-a} AK^a L^{1-a} = \lambda AK^a L^{1-a} = \lambda Y$$

(3) The Cobb-Douglas production function satisfies the Inada conditions:

$$\lim_{K \to 0} \frac{\partial Y}{\partial K} = \lim_{K \to 0} \alpha A \left( \frac{L}{K} \right)^{1-a} = \infty \quad \lim_{L \to 0} \frac{\partial Y}{\partial L} = \lim_{L \to 0} (1 - \alpha) A \left( \frac{K}{L} \right)^{a} = \infty$$

$$\lim_{K \to \infty} \frac{\partial Y}{\partial K} = \lim_{K \to \infty} \alpha A \left( \frac{L}{K} \right)^{1-a} = 0 \quad \lim_{L \to \infty} \frac{\partial Y}{\partial L} = \lim_{L \to \infty} (1 - \alpha) A \left( \frac{K}{L} \right)^{a} = 0$$

In per capita terms the Cobb-Douglas becomes:

$$Y = A \left( \frac{K}{L} \right)^a L = Ak^a L \quad \Rightarrow \quad y = Ak^a$$

By the neoclassical assumptions:

$$MPK_t = \frac{\partial [L_t f(k_t)]}{\partial K_t} = f''(k_t) = aAk^{a-1} > 0$$

$$\frac{\partial MPK_t}{\partial K_t} = \frac{\partial f''(k_t)}{\partial K_t} = \frac{1}{L_t} f'''(k_t) < 0 \Rightarrow f'''(k_t) = (a - 1)aK^{a-2} < 0$$
Capital accumulation:

\[ \dot{K} = I_t - \delta K_t \]

where \( I_t \) is investment, \( \delta \) is the depreciation rate and \( \dot{K} = dK_t/dt \).

Investment funding via savings:

\[ I_t = S_t \kappa \alpha t S_t = sY_t \]

where \( 0 < s < 1 \) is the marginal propensity to save. By the last two eqs:

\[ \dot{K} = sY_t - \delta K_t \]

In per capita terms

\[ \frac{\dot{K}}{L_t} = s \frac{Y_t}{L_t} - \delta \frac{K_t}{L_t} \Rightarrow \frac{\dot{K}}{L_t} = s f(k_t) - \delta k_t \]

\[ \dot{k} \equiv \frac{d}{dt} \left( \frac{K_t}{L_t} \right) = \frac{L_t \frac{dK_t}{dt} - K_t \frac{dL_t}{dt}}{L_t^2} \Rightarrow \dot{k} = \frac{\dot{K}}{L_t} - k_t \frac{L}{L_t} \]

\[ \Rightarrow \dot{k} = s f(k_t) \left( \delta + \frac{\dot{L}}{L} \right) k_t \]

where \( \frac{\dot{L}}{L} = n \) is the labour force growth rate. Then, the law of motion for the capital stock in per capita terms becomes:

\[ \dot{k} = s f(k_t) - (n + \delta) k_t \]

\textbf{(FUNDAMENTAL DIFFERENTIAL EQUATION FOR CAPITAL STOCK ACCUMULATION IN THE SIMPLE NEOCLASSICAL SOLOW-SWAN MODEL)}
Equilibrium in the Solow-Swan model

\[
\begin{align*}
\left[ s f(k_t) \right] &> \left[ (n+\delta)k_t \right] \Rightarrow k > 0 \quad \text{and} \quad \left[ s f(k_t) \right] < \left[ (n+\delta)k_t \right] \Rightarrow k < 0
\end{align*}
\]

Steady state \((k = 0)\): \(\Rightarrow s f (k_t) = (n + \delta)k_t\)

Explanation:
\(\frac{k}{k} = s \frac{f(k_t)}{k_t} - (n + \delta)\)

Dynamic analysis in the Solow-Swan model

Conclusion by the solution of the fundamental differential equation of capital accumulation:

THE STEADY-STATE OUTPUT IS CONSTANT (THE GROWTH RATE IS ZERO) DUE TO DIMINISHING RETURNS!
A rise in the savings rate

If the savings rate changes from \(s\) to \(s'\) the \([s f(k_i)]\) curve moves upwards.

\(\Rightarrow\) The steady state capita-labour ratio rises from \(\bar{k}\) to \(\bar{k}'\).

\(\Rightarrow\) Per capita income rises.

Implication: If two countries have the same characteristics (parameters) but one of them has a higher savings rate, then it will have a higher per capita income (though both will have a zero growth rate).

**Question:** How much should a country save to achieve maximum welfare?

**Criterion:** Maximize per capita consumption (a function of income ofcourse)

\[
\bar{c}(s) = f'(\bar{k}(s)) - (n + \delta)\bar{k}(s)
\]

\[
\frac{d\bar{c}}{ds} = [f'(\bar{k}) - (n + \delta)]\frac{d\bar{k}}{ds}
\]

Since \(\frac{d\bar{k}}{ds} > 0\), we focus on \([f'(\bar{k})-(n+\delta)]\). Maximum consumption is attained when:

\[
\frac{d\bar{c}}{ds} = 0 \Rightarrow f''(\bar{k}^*) = n + \delta
\]

“Golden Rule” of capital accumulation
A contrast: the $AK$ model

Production function:

$$Y_t = AK_t$$

$A_t$ : technological constant, so that $MPK$ is constant and equal to $A$ (i.e. the capital-output ratio is constant). In per capita terms:

$$\frac{Y_t}{L_t} = A \frac{K_t}{L_t} \Rightarrow y_t = Ak_t$$

The fundamental differential equation for capital accumulation becomes:

$$\frac{\dot{k}}{k} = sA - (n + \delta)$$

When $sA > n + \delta$ ⇒ the growth rates of capital and output are constant and positive
The basic structure of the model with exogenous technological progress

Aggregate production function:

\[ Y_t = F(K_t, A_t L_t) \]

* with \( \frac{A}{A} = g > 0 \)

Neoclassical assumptions are satisfied:

1. Positive and diminishing marginal returns of factor inputs:

2. Constant returns to scale w.r.t capital and labour:

\[ F(\lambda K_t, A_t (\lambda L_t)) = \lambda F(K_t, A_t L_t), \quad \lambda > 0 \]

3. Inada conditions

By assumption (2):

\[ \frac{Y_t}{A_t L_t} = F\left( \frac{K_t}{A_t L_t}, 1 \right) \Rightarrow \tilde{y}_t = f(\tilde{k}_t) \]

where \( \tilde{y}_t = Y_t / A_t L_t \) is output per effective unit of labour, \( \tilde{k}_t = K_t / A_t L_t \) and \( f(\tilde{k}_t) = F(\tilde{k}_t, 1) \).

Capital accumulation equation:

\[ \dot{K} = sY_t - \delta K_t \]

In terms of effective units of labour \( A_t L_t \):

\[ \frac{\dot{K}}{A_t L_t} = s \frac{Y_t}{A_t L_t} - \delta \frac{K_t}{A_t L_t} \Rightarrow \frac{\dot{K}}{A_t L_t} = s\tilde{y}_t - \delta \tilde{k}_t \]
By definition we have

\[
\dot{k} = -d\left(\frac{K_t}{A_tL_t}\right) = \frac{KA_tL_t - K_t(ALL_t + A_tL)}{(A_tL_t)^2} \implies \dot{k} = \frac{K}{A_tL_t} - \ddot{k}t \left(\frac{A_tL_t}{A_tL_t}\right)
\]

If \( \frac{L}{L} = n \), then \( \ddot{k} = \frac{K}{A_tL_t} - (g + n)\ddot{k}t \implies \)

\[
\ddot{k} = sf(\ddot{k}) - (n + g + \delta)\ddot{k}_t
\]

(FUNDAMENTAL DIFFERENTIAL EQUATION FOR CAPITAL STOCK ACCUMULATION IN THE NEOCLASSICAL SOLOW-SWAN MODEL WITH EXOGENOUS TECHNOLOGICAL PROGRESS

Equilibrium in the neoclassical model with exogenous technological progress

Steady state: \( \ddot{k} = 0 \implies sf(\ddot{k}) = (n + g + \delta)\ddot{k} \)

\[
\ddot{y} = f(\ddot{k}) = \frac{Y_t}{A_tL_t} = \frac{y_t}{A_t} = \text{constant} \implies \frac{\ddot{y}}{\dot{y}} = \frac{A_t}{A} = g.
\]
Speed of convergence

Around the steady-state \( \bar{k} \), the fundamental equation is given by a Taylor approximation:

\[
\hat{k} = \left( \frac{\partial \hat{k}(\bar{k})}{\partial \hat{k}_t} \right)_{\hat{k}_t = \bar{k}} (\bar{k}_t - \bar{k}) \Rightarrow \hat{k} \text{ depends upon the distance } (\bar{k}_t - \bar{k})
\]

The first term is given by:

\[
\frac{\partial \hat{k}}{\partial \hat{k}_t} = sf''(\bar{k}) - (n + g + \delta)
\]

Since \( s = \frac{(n + g + \delta)\bar{k}}{f(\bar{k})} \), we get:

\[
\frac{\partial \hat{k}(\bar{k})}{\partial \hat{k}_t} = \frac{(n + g + \delta)\bar{k}f'(\bar{k})}{f(\bar{k})} - (n + g + \delta) = [\varepsilon_{\text{YK}}(\bar{k}) - 1](n + g + \delta)
\]

where \( \varepsilon_{\text{YK}}(\bar{k}) = f'(\bar{k}) \frac{\bar{k}}{f(\bar{k})} \) is the elasticity of the production function w.r.t capital. Replacing in the Taylor approximation:

\[
\hat{k} = \left[ \varepsilon_{\text{YK}}(\bar{k}) - 1 \right](n + g + \delta) (\bar{k}_t - \bar{k})
\]

Solution:

\[
k_t - \bar{k} = e^{-[1-\varepsilon_{\text{YK}}(\bar{k})](n+g+\delta)t} (\bar{k}_0 - \bar{k})
\]

Conclusion: The rate at which \( \bar{k}_t \) approaches equilibrium is constant and equal to \([1 - \varepsilon_{\text{YK}}(\bar{k})](n + g + \delta)\).
Relaxing the two input constraint: human capital in the Solow model

Question: Will an increase in the number of production factors eliminate the impact of diminishing returns?

An obvious candidate: Human capital
(covers knowledge through education and/or experience, skills, ideas and anything else that can be accumulated by the individual and used in the production process –and is not covered under ‘physical’ capital)

A digression: Why do individuals accumulate human capital?

Again, an obvious answer:
- In order to gain from increased returns through the rise in the wage rate

... and some difficulties associated with human capital
- It is practically impossible to measure the human capital stock
- We can only proxy it (say, through the number of years of education)
- Utilizing its concept in the context of a growth model is necessarily vague

The basic structure of the Solow-Swan model with human capital

Aggregate production function:

\[ Y_t = F(K_t, H_t, L_t) \]

where \( H_t \) : human capital stock

Additional assumptions on the production function:

(1) positive and diminishing marginal products with respect to human capital:

\[
\frac{\partial Y_t}{\partial H_t} > 0 \quad \frac{\partial^2 Y_t}{\partial H_t^2} < 0
\]

(2) constant returns to scale:

\[ F(\lambda K_t, \lambda H_t, \lambda L_t) = \lambda F(K_t, H_t, L_t), \quad \lambda > 0 \]

(3) Inada conditions:

\[
\lim_{H_t \to 0} \frac{\partial Y_t}{\partial H_t} = \infty \quad \lim_{H_t \to \infty} \frac{\partial Y_t}{\partial H_t} = 0
\]
Mechanics of the model:

By constant returns to scale:

\[ \frac{Y_t}{L_t} = F\left( \frac{K_t}{L_t}, \frac{H_t}{L_t}, 1 \right) \Rightarrow y_t = f(k_i, h_i) \]

Capital accumulation:

\( \dot{K} = (I_K)_t - \delta_K K_t, \quad \dot{H} = (I_H)_t - \delta_H H_t \)

Investment financing:

\( (I_K)_t = s_K \gamma_t, (I_H)_t = s_H \gamma_t \quad 0 < s_K, s_H < 1, \quad s_K + s_H < 1 \)

Fundamental equations of the Solow-Swan model with human capital:

\[ \dot{k} = s_K f(k_t, h_t) - (n + \delta_K)k_t \]

\[ \dot{h} = s_H f(k_t, h_t) - (n + \delta_H)h_t \]

Equilibrium:

\[ k = 0 \quad \Rightarrow \quad s_K f(k_t, h_t) = (n + \delta_K)k_t \quad \text{will yield} \quad \bar{k} \]

\[ h = 0 \quad \Rightarrow \quad s_H f(k_t, h_t) = (n + \delta_H)h_t \quad \text{will yield} \quad \bar{h} \]

Result. The Solow-Swan model with human capital does not predict long-run economic growth.

Intuition: why do diminishing returns to physical and human capital lead to equilibrium where the essential variables of the model \((y, k, h)\) remain constant?

We have:

\[ \frac{\dot{k}}{k} = \frac{s_K f(k_t, h_t)}{k_t} - (n + \delta_K), \quad \frac{\dot{h}}{h} = \frac{s_H f(k_t, h_t)}{h_t} - (n + \delta_H) \]

If \( \frac{\dot{k}}{k} > 0 \quad \Rightarrow \quad s_K \frac{f(k_t, h_t)}{k_t} > (n + \delta_K) \), that cannot hold unless \( \frac{\dot{h}}{h} > \frac{\dot{k}}{k} > 0 \), which in turn would lead to a continuous decline in \( \frac{f(k_t, h_t)}{h_t} \)

\[ \Rightarrow \text{only} \quad \frac{\dot{h}}{h} = \frac{\dot{k}}{k} = 0 \quad \text{is the feasible equilibrium.} \]
CONVERGENCE IN THE NEOCLASSICAL MODEL

Definition of convergence: the path of the variable towards a specific value.

Various aspects of convergence can be examined:
National: convergence across countries
Domestic: convergence across regions
Global: convergence across groups of countries

Convergence in the Solow-Swan model

Cobb-Douglas production function: \( Y_i = AK_i^\alpha L_i^{1-\alpha} \)

Steady-state capital-labor ratio in this economy: \( \bar{k}_i = \left[ \frac{sA}{n + \delta} \right]^{\frac{1}{1-\alpha}} \)

Steady-state income per capita: \( \bar{y}_i = A^{\frac{1}{1-\alpha}} \left[ \frac{s}{n + \delta} \right]^{\frac{\alpha}{1-\alpha}} \)

Production function in per capita terms \( y_i = Ak_i^\alpha \)
\( \Rightarrow \) growth rate \( g_y \) of country \( i \): \( g_y = ag_k \)

At the steady state \( g_y = g_k = 0 \) with the capital-labor ratio \( k \) increases at a diminishing rate while approaching the steady-state point \( \bar{k} \) (if \( k(0) < \bar{k} \)).

Result. Two economies that have the same production function and the same values of the parameters \( s, n, \delta \) will exhibit the same the long-run equilibrium level of income per capita.

This results says that structurally similar economies will converge in the long run to the same income level.
Dynamics of the growth rate of $k$ along the transition path: 
\[ g_k = sA k^{-(1-a)} - (n + \delta) \]

Dynamics of the growth rate of $y$ along the transition path: 
\[ g_y = a sA k^{-(1-a)} - a(n + \delta) \]

\[ \Rightarrow \frac{\partial g_y}{\partial k} < 0, \text{ i.e. growth rate is inversely proportional to the capital-labor ratio along the economy’s transition to the steady-state} \]

(along the economy’s transition to the steady-state (the lower the initial capital-labor ratio, the higher the growth rate will be).

‘Absolute Convergence’ hypothesis: In the neoclassical model of exogenous growth, the economy tends to grow faster for a low level of initial capital-labor ratio compared to the case of a higher capital-labor ratio.

Explanation:
If capital exhibits diminishing returns ($\alpha < 1$), an economy with lower capital-labor ratio exhibits a higher marginal product of capital and thus, grows faster compared to a similar economy with a higher capital-labor ratio \( \Rightarrow \) the differences across countries will tend to fade out over time, with per capita income and its growth rate gradually converging until reaching an identical long-run equilibrium level for both countries, respectively.

Otherwise stated: if all economies have similar characteristics, the ‘poor’ (‘rich’) countries with low (high) initial capital $k_0$, and, thus, low (high) initial income $y_0$, will display a higher (lower) growth rate (the two variables are expected to be negatively correlated).

But: can it be always true that two economies with different levels of initial income will exhibit different (and inversely related) growth rates? (or, is it possible for a typical African country with low capital and income levels to grow faster than Switzerland?)

Answer: ‘NO!’
Recall: ‘Absolute Convergence’ applies to economies with identical structural parameters.

(If, for example, if the population growth rate $n$ is higher in a country, then it will exhibit lower steady-state income and growth rate along transition to equilibrium than the latter.)

$\Rightarrow$ Convergence should only be anticipated across structurally ‘similar’ economies (i.e. with identical parameter values).

‘Conditional Convergence’ hypothesis: In the neoclassical model of exogenous growth, economies tend to converge faster to their own steady state the further away they are from it.

An economy is likely to exhibit both a high initial income and fast growth simply because its steady-state income is simultaneously high and relatively far from its current income.

Example: large values of the technology constant $A$ result in high steady-state capital-labor ratio and income.

$\Rightarrow$ A simple comparison with an economy having the same initial capital stock and income level without looking at technology (parameter $A$) would lead to inaccurate conclusions regarding its convergence pattern.
Empirical tests on convergence

- General formulation: $g_{y_{i,t}, t+T} = f[\log(y_{i,t})]$

$g_{y_{i,t}, t+T} = \log(y_{i,t+T} / y_{i,t}) / T$ : average growth rate of economy $i$ from $t$ to $t+T$

$\log(y_{i,t})$ : log of income of economy $i$ in period $t$

Cobb-Douglas production function: linear approximation around the steady-state yields of the capital growth rate (which determines the income growth rate):

$$g_k = \frac{d \log(K)}{dt} \approx -(1 - \alpha)(n + \delta) \log(\frac{k}{K})$$

Using $\log(\frac{y}{y}) = a \log(\frac{k}{k})$, the income growth rate is: $g_y \approx -(1 - \alpha)(n + \delta) \log(\frac{y}{y})$

- General specification: $g_y = -\beta \log(\frac{y}{y})$, where $\beta = (1 - \alpha)(n + \delta)$.

Convergence is determined by $\beta$: for $\beta > 0$, if the initial income is lower (higher) than the steady-state income, the economy will grow faster (more slowly) $\Rightarrow \beta$-convergence.

Convergence speed: coefficient $\beta$ shows how fast the economies approach their long-run equilibrium: higher (lower) $\beta$ implies faster (lower) convergence.

Solving the last equation yields: $\log(y_t) = (1 - e^{-\beta T}) \log(\bar{y}) + e^{-\beta T} \log(y_0)$

- Income at any time $t$ depends on the constant value of the initial income $y_0$ and of the steady-state income $\bar{y}$.
- For higher values of $\beta$, income $y_t$ will be closer to its steady-state level.

At $t=T$: $\frac{\log(\frac{y_T}{y_0})}{T} = \frac{(1 - e^{-\beta T})}{T} \log(\frac{\bar{y}}{y_0})$

The average growth rate after $T$ periods is inversely related to the ratio of the initial and the steady-state income.
‘Absolute’ and ‘Conditional’ β-convergence: In the neoclassical model of exogenous growth, absolute β-convergence is measured in terms of the initial income of the economy, whereas ‘conditional’ β-convergence is given by the deviation of the initial income level from its steady-state level.

Question: If each economy tends to converge to its steady-state income level, will convergence emerge at the global level, i.e. will the inequality of income distribution be worldwide diminished?

Intuition: for a small income gap across economies worldwide, individual economies should tend to converge to equilibrium income per capita.

But: does β-convergence lead necessarily to small income dispersion?

Answer: ‘NO!’: This income gap can stay constant (or increase) despite the presence of β-convergence.

Example: a poor country may grow much faster than a rich one so that the income of the former may turn out to exceed the level of the latter and thus the income gap will increase while there is β-convergence in both countries (Galton’s Fallacy).

Consider the case of successive time periods (T=1) between t-1 and t:

\[
\log(y_t / y_{t-1}) = (1 - e^{-\beta}) \log(\bar{y}) - (1 - e^{-\beta}) \log(y_{t-1}) \Leftrightarrow \log(y_t) = e^{-\beta} \log(y_{t-1}) + c
\]

where \( c = (1 - e^{-\beta}) \log \bar{y} \). Income dispersion is then given by the variance:

\[
Var[\log(y_t)] = e^{-2\beta} Var[\log(y_{t-1})]
\]

Setting \( Var[\log(y_t)] = \sigma_t^2 \):

\[
\sigma_t^2 = e^{-2\beta} \sigma_{t-1}^2
\]

\( \Rightarrow \) in order for income dispersion to be reduced intertemporally we must have that \( \beta > 0 \) (β-convergence), in which case \( \sigma_t^2 < \sigma_{t-1}^2 \) (σ-convergence)

Relation between β-convergence and σ-convergence: In the neoclassical model of exogenous growth β-convergence is a necessary but not sufficient condition for σ-convergence.
Log of real per capital income and growth rate in 119 countries


Log of real per capital income and growth rate in OECD countries


Intertemporal dispersion of log of real per capital income in all economies and OECD countries